Review: Derivatives of Logarithms - 11/2/16

1 Derivative of Natural Log

Example 1.0.1 *We want to find* $\frac{d}{dx}\ln(x)$. *Let* $\ln(x) = y$. *Then* $e^y = x$. *Let's use implicit differentiation on this. Taking the derivative of both sides gives us* $e^y \frac{dy}{dx} = 1\frac{dx}{dx}$, *so* $\frac{dy}{dx} = \frac{1}{e^y}$. *But we already know that* $e^y = x$, *so*

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Note: All points in the domain of the derivative must also be in the domain of the original function. Thus the domain of $\frac{d}{dx}\ln(x)$ is $(0, \infty)$. It does NOT include negative numbers!

Example 1.0.2 *What is* $\frac{d}{dx}\frac{\ln(x)}{3x}$? *We can use the quotient rule: let* $f(x) = \ln(x)$ *and* $g(x) = 3x$, *then* $f'(x) = \frac{1}{x}$ *and* $g'(x) = 3$. *Then* $\frac{d}{dx}\frac{\ln(x)}{3x} = \frac{\frac{1}{x}3x - 3\ln(x)}{(3x)^2} = \frac{3-3\ln(x)}{9x^2} = \frac{1-\ln(x)}{3x^2}$.

Example 1.0.3 *What is* $\frac{d}{dx}\ln(x^2 - 3)$? *We need to use the chain rule: let* $f(u) = \ln(u)$ *and* $g(x) = x^2 - 3$, *so* $f'(x) = \frac{1}{x}$ *and* $g'(x) = 2x$. *Then* $\frac{d}{dx}\ln(x^2 - 3) = \frac{1}{x^2-3} \cdot 2x$. *The domain of the original function is* $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$. *What about the domain for the derivative? If this were just an ordinary function, then the domain would be all points except* $x = \sqrt{3}$. *However, it's a derivative, so it can't have any points in its domain that aren't in the domain of the original function. Thus the domain for the derivative is also* $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

2 Derivative of Logarithms

Example 2.0.4 *Find* $\frac{d}{dx}\log_b(x)$. *Note that we can rewrite this as* $\frac{d}{dx}\frac{\ln(x)}{\ln(b)}$ *using the change of base formula. Since* $\ln(b)$ *is a number, we can pull this outside using the constant multiple rule, so we have* $\frac{1}{\ln(b)}\frac{d}{dx}\ln(x) = \frac{1}{x\ln(b)}$.

From the above calculations, we have found:

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$

Example 2.0.5 *Find* $\frac{d}{dx}\log_2(x)$. *Here we just substitute in 2 for b to get* $\frac{d}{dx}\log_2(x) = \frac{1}{x\ln(2)}$.

Example 2.0.6 *Find* $\frac{d}{dx}\log_3(x^2 - 5)$. *We can use the chain rule: let* $f(u) = \log_3(u)$ *and* $g(x) = x^2 - 5$, *so* $f'(u) = \frac{1}{u\ln(3)}$ *and* $g'(x) = 2x$. *Then* $\frac{d}{dx}\log_3(x^2 - 5) = \frac{1}{(x^2-5)\ln(3)} \cdot 2x$.

3 Derivative of Exponential Functions

Example 3.0.7 *Find* $\frac{d}{dx}b^x$*. Recall that* $b = e^{\ln(b)}$ *since* $\ln$ *is the inverse function of* e*. Then we can rewrite this as* $\frac{d}{dx}(e^{\ln(b)})^x = \frac{d}{dx}e^{(\ln(b))x}$*. Now we can use the chain rule to find the derivative: let* $f(u) = e^u$ *and* $g(x) = \ln(b)x$*, so* $f'(u) = e^u$ *and* $g'(x) = \ln(b)$*. Then* $\frac{d}{dx}b^x = \frac{d}{dx}e^{(\ln(b))x} = e^{\ln(b)x} \cdot \ln(b) = b^x \ln(b)$*.*

From the above calculations, we have found:

$$\frac{d}{dx}b^x = b^x \ln(b).$$

Example 3.0.8 *Find* $\frac{d}{dx}2^x$*. We just substitute 2 in for* b *to get* $\frac{d}{dx}2^x = 2^x \ln(2)$*.*

Example 3.0.9 *Find* $\frac{d}{dx}2^x \arcsin(x)$*. We can use the product rule: let* $f(x) = 2^x$ *and* $g(x) = \arcsin(x)$*, so* $f'(x) = 2^x \ln(x)$ *and* $g'(x) = \frac{1}{\sqrt{1-x^2}}$*. Then* $\frac{d}{dx}2^x \arcsin(x) = 2^x \ln(x) \arcsin(x) + \frac{2^x}{\sqrt{1-x^2}}$*.*

Example 3.0.10 *Find* $\frac{d}{dx}5^{\frac{x+2}{3x^2}}$*. We first use the chain rule: let* $f(u) = 5^u$ *and* $g(x) = \frac{x+2}{3x^2}$*, so* $f'(u) = 5^u \ln(5)$*. To find the derivative of* g*, we need the quotient rule: let* $z(x) = x + 2$ *and* $q(x) = 3x^2$*, so* $z'(x) = 1$ *and* $q'(x) = 6x$*. Then* $g'(x) = \frac{3x^2-6x(x+2)}{(3x^2)^2}$*. Then* $\frac{d}{dx}5^{\frac{x+2}{3x^2}} = 5^{\frac{x+2}{3x^2}} \ln(5) \cdot \frac{3x^2-6x(x+2)}{(3x^2)^2}$*.*

Practice Problems

1. Find $\frac{d}{dx}\sin(\ln(x))$.
2. Find $\frac{d}{dx}\log_3(xe^x)$.
3. Find $\frac{d}{dx}\log_3(2^x)$.
4. Let $\log_2(x + y) = y^2$. Find $\frac{dy}{dx}$.

Solutions

1. We can use the chain rule to get $\frac{d}{dx}\sin(\ln(x)) = \cos(\ln(x)) \cdot \frac{1}{x}$.
2. We can use the chain rule to get $\frac{d}{dx}\log_3(xe^x) = \frac{1}{xe^x\ln(3)} \cdot (e^x + xe^x)$.
3. We can use the chain rule to get $\frac{d}{dx}\log_3(2^x) = \frac{1}{2^x\ln(3)} \cdot 2^x\ln(2) = \frac{\ln(2)}{\ln(3)}$.
4. We can use implicit differentiation. First, find the derivative of both sides: $\frac{1}{(x+y)\ln(2)} \cdot (y\frac{dx}{dx} + x\frac{dy}{dx}) = 2y\frac{dy}{dx}$. Now we move $\frac{dy}{dx}$ to one side: $\frac{y}{(x+y)\ln(2)} = 2y\frac{dy}{dx} - \frac{x}{(x+y)\ln(2)}\frac{dy}{dx}$. Now we solve for $\frac{dy}{dx}$ to get
$$\frac{dy}{dx} = \frac{\frac{y}{(x+y)\ln(2)}}{2y - \frac{x}{(x+y)\ln(2)}} = \frac{y}{2y(x+y)\ln(2) - x}.$$